RESULTANT OF AN EQUIVARIANT POLYNOMIAL SYSTEM WITH RESPECT TO THE SYMMETRIC GROUP

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The analysis and solving of polynomial systems are fundamental problems in computational algebra. In many applications, polynomial systems are highly structured and it is very useful to develop specific methods in order to take into account a particular structure. In this talk, we will focus on systems of n homogeneous polynomials $f_1,\ldots,f_n$ in n variables $x_1,\ldots,x_n$ that are globally invariant under the action of the symmetric group S_n of n symbols. More precisely, we will assume that for any integer $i \in \{1,2,\ldots,n\}$ and any permutation $\sigma \in S_n$

$$\sigma(f_i) := f_i(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) = f_{\sigma(i)}(x_1,x_2,\ldots,x_n). \tag{1}$$

In the language of invariant theory these systems are called equivariant with respect to the symmetric group S_n, or simply S_n-equivariant (see for instance [5, §4] or [1, Chapter 1]). Some recent interesting developments based on Gröbner basis techniques for this kind of systems can be found in [2] with applications. In this work, we will study the resultant of these systems.

The main result of this talk is a decomposition of the resultant of a S_n-equivariant polynomial system. This formula allows to split such a resultant into several other resultants that are in principle easier to compute and that are expressed in terms of the divided differences of the input polynomial system. We emphasize that the multiplicity of each factor appearing in this decomposition is also given. Another important point of our result is that it is an exact and universal formula which is valid over the universal ring of coefficients (over the integers) of the input polynomial system. Indeed, we payed attention to use a correct and universal definition of the resultant. In this way, the formula we obtain has the correct geometric meaning and stays valid over any coefficient ring by specialization. This kind of property is particularly important for applications in the fields of number theory and arithmetic geometry where the value of the resultant is as important as its vanishing.

The discriminant of a homogeneous polynomial is also a fundamental tool in Computer Algebra. Although the discriminant of the generic homogeneous polynomial of a given degree is irreducible, for a particular class of polynomials it can be decomposed and this decomposition is always deeply connected to the geometric properties of this class of polynomials. The second main Theorem of this talk is a decomposition of the discriminant of a homogeneous symmetric polynomial. This result was actually the first goal of this work that has been inspired by the unpublished (as far as we know) note [4] by N. Perminov and S. Shakirov where a first tentative for such a formula is given without a complete proof. Another motivation was also to improve the computations of discriminants for applications in convex geometry, following a paper by J. Nie where the boundary of the cone of non-negative polynomials on an algebraic variety is studied by means of discriminants [3]. We emphasize that our formula is obtained as a byproduct of our first formula on the resultant of a S_n-equivariant polynomial system. Therefore, it inherits from the same features, namely it allows to split a discriminant into several resultants that are easier to compute and it is a universal formula where the multiplicities of the factors are provided. Here again, we payed attention to use a correct and universal definition of the discriminant.

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